# INSTANTS OF SMALL AMPLITUDE OF BROWNIAN MOTION AND APPLICATION TO THE KUBILIUS MODEL

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ABSTRACT. Let $W(t), t \geq 0$ be standard Brownian motion. We study the size of the time intervals $I$ which are admissible for the long range of slow increase, namely given a real $z > 0$,

$$\sup_{t \in I} \frac{|W(t)|}{\sqrt{t}} \leq z,$$

and we estimate their number of occurences. We obtain optimal results in terms of class test functions and, by means of the quantitative Borel-Cantelli lemma, a fine frequency result concerning their occurences. Using Sakhanenko's invariance principe to transfer the results to the Kubilius model, we derive applications to the prime number divisor function. We obtain refinements of some results recently proved by Ford and Tenenbaum in [4].

## 1. Introduction-Main results

Let $W(t), t \geq 0$ be standard Brownian motion. Let $z$ be some positive real. The study of the number of occurences of the time intervals $I$ for which

$$\sup_{t \in I} \frac{|W(t)|}{\sqrt{t}} \leq z,$$

is the first motivation of this work. In a second step, we will derive applications for the Kubilius model in number theory. More precisely, let $f : [1, \infty) \to \mathbb{R}^+$ be here and throughout a non-decreasing function such that $f(t) \uparrow \infty$ with $t$ and

$$f(t) = o_\rho(t^\rho). \tag{1.1}$$

We will consider intervals of type $I = [N, Nf(N)]$. We essentially examine the case $N = e^k, \ k = 1, 2, \ldots$. The study made can be extended with no difficulty to more general geometrically increasing sequences, but this aspect will be not developed. Put

$$A_k(f, z) = \left\{ \sup_{e^k \leq t \leq e^k f(e^k)} \frac{|W(t)|}{\sqrt{t}} < z \right\}, \qquad k = 1, 2, \ldots \tag{1.2}$$

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Let $U(t) = W(e^t)e^{-t/2}, t \in \mathbb{R}$ be the Ornstein-Uhlenbeck process. It will be more convenient to work with $U$ instead of $W$. Observe that

$$A_k(f, z) = \Big\{ \sup_{k \leq s \leq k + \log f(e^k)} |U(s)| \leq z \Big\}.$$

And so as $U$ is stationary

$$\mathbb{P}\{A_k(f, z)\} = \mathbb{P}\Big\{ \sup_{0 \leq s \leq \log f(e^k)} |U(s)| \leq z \Big\}.$$

We say that $f \in \mathcal{U}_z$ whenever $\mathbb{P}\{ \limsup_{k \to \infty} A_k(f, z)\} = 0$, and that $f \in \mathcal{V}_z$ if $\mathbb{P}\{ \limsup_{k \to \infty} A_k(f, z)\} = 1$. By the 0-1 law (since $U$ is strongly mixing), the latter probabilities can only be 0 or 1.

Notice that if $f \in \mathcal{U}_z$, then with probability one

$$J(f) := \liminf_{k \to \infty} \sup_{k \leq s \leq k + \log f(e^k)} |U(s)| > z,$$

whereas $J(f) \leq z$, almost surely if $f \in \mathcal{V}_z$. In the latter case, it makes sense to estimate the size of the counting function

$$N_n(f, z) = \sum_{k=1}^{n} \chi_{A_k(f, z)} \qquad n = 1, 2, \dots$$

Naturally this has to be done with respect to the corresponding means $\nu_n(f, z) := \mathbb{E} N_n(f, z)$.

We shall first characterize the classes $\mathcal{U}_z$ and $\mathcal{V}_z$ by means of a simple convergence criterion, and complete our characterization by including a frequency result concerning the class $\mathcal{V}_z$.

**Theorem 1.1.** *There exists $\lambda(z) > 0$ with $\lambda(z) \sim \frac{\pi^2}{4z^2}$ as $z \to 0$, such that if $\Sigma(f) = \sum_k f(e^k)^{-\lambda(z)}$, then*

$$f \in \mathcal{U}_z \quad (resp. \in \mathcal{V}_z) \quad \Longleftrightarrow \quad \Sigma(f) < \infty \quad (resp. = \infty).$$

*Further for any $a > 3/2$,*

$$N_n(f, z) \stackrel{a.s.}{=} \nu_n(f, z) + \mathcal{O}\big(\nu_n^{1/2}(f, z) \log^a \nu_n(f, z)\big).$$

*And there are positive constants $K_1(z), K_2(z)$ depending on $z$ only, such that for all $n$*

$$K_1(z) \leq \frac{\nu_n(f, z)}{\sum_{k=1}^{n} f(e^k)^{-\lambda(z)}} \leq K_2(z).$$

The critical value $\lambda(z)$ is the smallest eigenvalue in the Sturm-Liouville equation (2.1). See section 2.

The class of functions $f_c(t) = \log^c t$, $c > 0$, is of special interest in view of applications to the Kubilius model. We deduce from Theorem 1.1:

**Corollary 1.2.** *If $c > 1/\lambda(z)$, then $f_c \in \mathcal{U}_z$ whereas $f_c \in \mathcal{V}_z$ if $0 < c \leq 1/\lambda(z)$. Further, for any $0 < c \leq 1/\lambda(z)$ and $a > 3/2$,*

$$N_n(f_c, z) \stackrel{a.s.}{=} \nu_n(f_c, z) + \mathcal{O}\Big(\nu_n^{1/2}(f_c, z) \log^a \nu_n(f_c, z)\Big).$$

*And for all $n$*

$$K_1(z) \leq \frac{\nu_n(f_c, z)}{\sum_{k=1}^n k^{-c\lambda(z)}} \leq K_2(z).$$

Accordingly, if

$$I(f) := \liminf_{k \to \infty} \sup_{e^k \leq t \leq e^k f(e^k)} \frac{|W(t)|}{\sqrt{t}}, \tag{1.3}$$

then $\mathbb{P}\{I(f_c) \leq z\} = 1$ if and only if $0 < c \leq 1/\lambda(z)$. This is clear in view of (1.2). Noticing that $I(f) \leq I(g)$ whenever $f(N) \leq g(N)$ for all $N$ large, we therefore also deduce

**Corollary 1.3.** *We have $\mathbb{P}\{I(f_c) \leq z\} = 1$ if and only if $0 < c \leq 1/\lambda(z)$. And $\mathbb{P}\{I(f) = \infty\} = 1$ if $f(t) \gg_c f_c(t)$ for all $c$.*

*Remark* 1.4. This slightly improves upon Theorem 3 in [4], where it was shown that $\mathbb{P}\{I(f) < \infty\} = 1$ if $f(N) = (\log N)^b$ for some $b > 0$, whereas $\mathbb{P}\{I(f) = \infty\} = 1$, if $f(N) = (\log N)^{b(N)}$ with $b(N) \to \infty$ with $N$.

In [4], the behavior of corresponding functionals $I_f$ for sums of independent random variables (assuming only second absolute moments) was also considered. In this direction, we will also establish the following result for sums of independent random variables.

**Theorem 1.5.** *Let $\{X_j, j \geq 1\}$ be independent centered random variables. Assume that for some $\alpha > 2$,*

$$\sum_{j \geq 1} \mathbb{E}|X_j|^\alpha = \infty \qquad and \qquad v = \sup_{j \geq 1} \frac{\mathbb{E}|X_j|^\alpha}{\mathbb{E}|X_j|^2} < \infty. \tag{1.4}$$

*Let $Z_n = X_1 + \ldots + X_n$, $z_n^2 = \mathbb{E}Z_n^2$, $J_n = \{j : n \leq z_j^2 \leq nf(n)\}$. Then there exists a Brownian motion $W$ such that*

$$\liminf_{k \to \infty} \sup_{e^k \leq z_j^2 \leq e^k f(e^k)} \frac{|Z_j|}{z_j} \stackrel{a.s.}{=} \liminf_{k \to \infty} \sup_{e^k \leq z_j^2 \leq e^k f(e^k)} \frac{|W(z_j^2)|}{s_j}.$$

*almost surely. In particular if $c < 1/\lambda(z)$, then*

$$\liminf_{k \to \infty} \sup_{e^k \leq z_j^2 \leq e^k f_c(e^k)} \frac{|Z_j|}{z_j} \leq z, \qquad almost\ surely.$$

Notice in the iid case that assumption (1.4) simply reduces to the integrability condition $\mathbb{E}|X_1|^\alpha < \infty$ for some $\alpha > 2$.

Now introduce the truncated prime divisor function $\omega(m, t) = \#\{p \leq t : p|m\}$. Here and throughout we reserve the letter $p$ to denote some arbitrary prime number. Put

$$\rho(m, t) := \frac{|\omega(m, t) - \log \log t|}{\sqrt{\log \log t}}. \tag{1.5}$$

The local variations of $\rho(m, t)$ were recently investigated by Ford and Tenenbaum, who obtained in [4], after a careful study of the size of intervals of slow growth for general sums of independent random variables, quite elaborated asymptotic

estimates, on the basis of the approximation formula (2.7). The results concern the functional

$$\max_{N\leq\log\log t\leq Nf(N)}\rho(m,t). \tag{1.6}$$

Let $f(m)$, $g(m)$ be increasing and tending to infinity with $m$. It is notably proved ([4], Theorem 5) that if $g(m) \leq (\log\log m)^{1/10}$ and $f(N) = (\log N)^{\xi(N)}$ where $\xi(N) \to \infty$ sufficiently slowly so that $f(N) \leq N$, then

$$\min_{\substack{g(m)\leq N \\ Nf(N)\leq\log\log m}} \max_{N\leq\log\log t\leq Nf(N)} \rho(m,t) \to \infty \tag{1.7}$$

along a set of integers $m$ of natural density 1.

Further, if $f(N) = (\log N)^c$ and $g^2(m)(\log g^2(m))^c \leq \log\log m$ for $m$ large, then on a set of integers $m$ of density 1, we have

$$\min_{g(m)\leq N\leq g^2(m)} \max_{N\leq\log\log t\leq Nf(N)} \rho(m,t) \leq 30\sqrt{1+c}. \tag{1.8}$$

This provides informations on the size of intervals which are admissible for the long range of slow increase, in terms of the natural density on the integers. For instance $g(m) = \sqrt{(\log\log m)/(\log\log\log m)^c}$ is suitable. The principle followed in the proofs consists with modifying the proofs of the preliminary results on the size of intervals of slow growth for sums of independent random variables for the particular sequence $\{T_n, n \geq 1\}$ (section 2) and next to apply approximation formula (2.7).

Here we will proceed slightly differently. As we have optimal results on instants of small amplitude of Brownian motion, we directly compare the functionals (1.6) with analogous functionals of Brownian motion by means of Sakhanenko's invariance principle (Lemma 2.4). This is done in Theorem 1.6 below. This allows to transfer our previous results to truncated prime divisor function, not fully naturally, but sufficiently much to get new quite sharp results. More precisely, let $0 < M_1(x) < M_2(x)$, $M_2(x) \uparrow \infty$ with $x$. The previous results, as well as Theorem 1.1, Corollary 1.2, suggest to study the behavior for $x$ large, of the averages

$$\frac{1}{x}\#\Big\{m \leq x : \inf_{M_1(x)<N\leq M_2(x)} \sup_{N\leq s_j^2\leq Nf(N)} \frac{|\omega(m,j)-s_j^2|}{s_j} \leq z\Big\}. \tag{1.9}$$

Here we set

$$s_j^2 := \sum_{p\leq j}\frac{1}{p} - \frac{1}{p^2} = \log\log j + \mathcal{O}(1), \tag{1.10}$$

and the last relation comes from Mertens estimate. For technical reasons (scale invariance properties of $W$ and Kubilius model, see next section), it turns up that it is more convenient to replace the "$\log\log j$" term appeared before by $s_j^2$. The resulting modifications are thus neglectable in the statements. We show that the asymptotic order of the averages (1.9) can be quantified by using Ornstein-Uhlenbeck process. More precisely, let

$$I_N = I_N(f) = \big\{j : N \leq s_j^2 \leq Nf(N)\big\}. \tag{1.11}$$

Let also $\mathcal{N}$ denote some increasing sequence of positive reals tending to infinity. The theorem below allows to reduce the study of the averages (1.9) to the one of

similar questions for the Ornstein-Uhlenbeck process. Other formulations may be easily extrapolated from the proof.

**Theorem 1.6.** *Assume that* $M_2(x) = \mathcal{O}_\varepsilon(x^\varepsilon)$. *Let* $0 < z'' < z < z'$. *As* $x$ *tends to infinity,*

$$\mathbb{P}\Big\{\inf_{\substack{M_1(x)<N\leq M_2(x)\\N\in\mathcal{N}}} \sup_{j\in I_N} \frac{|W(s_j^2)|}{s_j} \leq z''\Big\} + o(1) \leq$$

$$\frac{1}{x}\#\Big\{m \leq x : \inf_{\substack{M_1(x)<N\leq M_2(x)\\N\in\mathcal{N}}} \sup_{j\in I_N} \frac{|\omega(m,j) - s_j^2|}{s_j} \leq z\Big\}$$

$$\leq \mathbb{P}\Big\{\inf_{\substack{M_1(x)<N\leq M_2(x)\\N\in\mathcal{N}}} \sup_{j\in I_N} \frac{|W(s_j^2)|}{s_j} \leq z'\Big\} + o(1).$$

By combining with Corollary 1.2, we deduce for instance

**Corollary 1.7.** *Assume that* $M_2(x) = \mathcal{O}_\varepsilon(x^\varepsilon)$ *and* $\log M_1(x) = o\big(\log M_2(x)\big)$. *Let* $c < 1/\lambda(z)$. *Then*

$$\lim_{x\to\infty} \frac{1}{x}\#\Big\{1 \leq m \leq x : \inf_{\log M_1(x)<k\leq\log M_2(x)} \sup_{e^k\leq s_j^2\leq e^k k^c} \frac{|\omega(m,j) - s_j^2|}{s_j} \leq z\Big\} = 1.$$

*Remark* 1.8. Let $d > 1$. There is no loss when restricting to $x^{1/d} \leq m \leq x$ in the above ratios. But $M_1(x) < e^k \leq M_2(x)$ imply $M_1(m) < e^k \leq M_2(m^d)$. This allows to deduce from Corollary 1.7 a result similar to those in [4] previously described, namely for any $d > 1$,

$$\lim_{x\to\infty} \frac{1}{x}\#\Big\{1 \leq m \leq x : \inf_{M_1(m)<e^k\leq M_2(m^d)} \sup_{e^k\leq s_j^2\leq e^k k^c} \frac{|\omega(m,j) - s_j^2|}{s_j} \leq z\Big\} = 1,$$

Taking for instance $M_2(x) = \log x$, $M_1(x) = \log^{\varepsilon(x)} x$, $\varepsilon(x) \to 0$, we get

$$\lim_{x\to\infty} \frac{1}{x}\#\Big\{1 \leq m \leq x : \inf_{(\log m)^{\varepsilon(m)}\leq e^k\leq\log m} \sup_{e^k\leq s_j^2\leq e^k k^c} \frac{|\omega(m,j) - s_j^2|}{s_j} \leq z\Big\} = 1.$$

$$(1.12)$$

And the relation $c \leq 1/\lambda(z)$ asymptotically becomes $\pi\sqrt{c}/2 \leq z$, $z \to 0$.

*Remark* 1.9. If instead of condition $\log M_1(x) = o\big(\log M_2(x)\big)$, we have the weaker assumption $\log M_1(x) = \rho \log M_2(x)$, $0 < \rho < 1$, then by operating similarly and using 0–1 law, we would also get for $\rho$ sufficiently small

$$\lim_{x\to\infty} \frac{1}{x}\#\Big\{1 \leq m \leq x : \inf_{M_2(m)^\rho\leq e^k\leq M_2(m)} \sup_{e^k\leq s_j^2\leq e^k k^c} \frac{|\omega(m,j) - s_j^2|}{s_j} \leq z\Big\} = 1.$$

$$(1.13)$$

However, we have no idea about a suitable precise value of $\rho$.

We will further establish a delicate frequency result for the truncated divisor function, which is in the spirit of Theorem 1.1.

**Theorem 1.10.** *Let $0 \leq M_1(x) < M_2(x)$, $M_1(x) \uparrow \infty$ such that $M(x) = \mathcal{O}_\varepsilon(x^\varepsilon)$. For any $z' > z > 0$ and $c < 1/\lambda(z)$, there exists a constant $\kappa > 0$ such that,*

$$\lim_{x \to \infty} \frac{1}{x} \# \left\{ m \leq x : \inf_{M_1(x) \leq n \leq M_2(x)} \frac{\#\left\{ k \leq n : \sup_{e^k \leq s_j^2 \leq e^k k^c} \frac{|\omega(m,j) - s_j^2|}{s_j} \leq z' \right\}}{n^{1 - c\lambda(z)}} \geq \kappa \right\}$$
$$= 1.$$

## 2. Auxiliary results

We first list the needed probabilistic results. Next we briefly describe Kubilius model and extract from the fundamental inequality a useful lemma. The underlying small deviation problem, namely the study for small $z$ of

$$\mathbb{P}\left\{ \sup_{0 \leq s \leq t} |U(s)| < z \right\},$$

can be yield to be intimately linked to the Sturm-Liouville equation

$$\psi''(x) - x\psi'(x) = -\lambda\psi(x), \qquad \psi(-z) = \psi(z) = 0. \tag{2.1}$$

Let $\lambda_1 \leq \lambda_2 \leq \ldots$ and $\psi_1(x), \psi_2(x), \ldots$ respectively denote the eigenvalues and normed eigenfunctions of Equation (2.1). Here $\lambda_i, \psi_j$ depend on $z$ and it is known that $\psi_1, \psi_2, \ldots$, form an orthonormal sequence with respect to the weight function $e^{-x^2/2}$. According to Newell's result (see [7] , see also (3.16) in [1])

$$\mathbb{P}\left\{ \sup_{0 \leq s \leq t} |U(s)| < z \right\} = \frac{1}{(2\pi)^{1/2}} \sum_{k=1}^{\infty} e^{-\lambda_k t} \left( \int_{-z}^{z} \psi_k(x) e^{-x^2/2} x \right)^2. \tag{2.2}$$

Let $\lambda(z)$ denote the smallest eigenvalue ($\lambda(z) = \lambda_1$). Then $\lambda(z) > 0$ is a strictly decreasing continuous function of $z$ on $(0, \infty)$. Further

$$\lambda(z) \sim \frac{\pi^2}{4z^2} \qquad \text{as } z \to 0. \tag{2.3}$$

See Lemma 3.1 in [1], see also Lemma 2.2 for the following result.

**Lemma 2.1.** (**Csáki's estimate**) *For $z > 0$, $t > 0$ we have*

$$\frac{e^{-\lambda(z)t}}{(2\pi)^{1/2}} \left( \int_{-z}^{z} \psi_1(x) e^{-x^2/2} x \right)^2 \leq \mathbb{P}\left\{ \sup_{0 \leq s \leq t} |U(s)| < z \right\} \leq \frac{e^{-\lambda(z)t}}{1 - e^{-t}}.$$

It follows that for $z > 0$, there exist positive constants $K_1(z), K_2(z)$ such that for all $t \geq 1$

$$K_1(z) e^{-\lambda(z)t} \leq \mathbb{P}\left\{ \sup_{0 \leq s \leq t} |U(s)| < z \right\} \leq K_1(z) e^{-\lambda(z)t}. \tag{2.4}$$

Now let $\mathcal{E}_s^t$ denote the vector space generated $U(u)$, $s \leq u \leq t$ and introduce the maximal correlation coefficient

$$\rho(\tau) = \sup_{\substack{\xi \in \mathcal{E}_{-\infty}^t \\ \eta \in \mathcal{E}_{t+\tau}^\infty}} \frac{|\mathbb{E}(\xi - \mathbb{E}\xi)(\eta - \mathbb{E}\eta)|}{\left[ |\mathbb{E}(\xi - \mathbb{E}\xi)^2 \mathbb{E}(\eta - \mathbb{E}\eta)^2| \right]^{1/2}}. \tag{2.5}$$

By stationarity, this one does not depend on $t$. Stationary Gaussian processes such that $\rho(\tau) \to 0$ as $\tau \to \infty$ are called completely regular. The spectral density of $U$ has the form $|\Gamma(\lambda)|^{-2}$ with $\Gamma(\lambda) = 1 + i\lambda$, which is obviously an entire function. Moreover, we also have $\frac{\log|\Gamma(\lambda)|}{1+\lambda^2} \in L^1(\mathbb{R})$. Further $\Gamma$ has $i$ as unique imaginary zero. As $\Im\left(\frac{1}{\lambda-i}\right) = \frac{1}{1+\lambda^2}$, it follows that $\sup_{\lambda\in\mathbb{R}}\left|\Im\left(\frac{1}{\lambda-i}\right)\right| < \infty$. Our next lemma is therefore just a direct consequence of Theorem 6, section VI.6 in [5].

**Lemma 2.2.** *The process $U$ is completely regular, and further*

$$\rho(\tau) = \mathcal{O}_\varepsilon(e^{-(1-\varepsilon)\tau}).$$

This result, which is due to Kolmogorov and Rozanov ([6], see Theorem 1 and remarks at end of p.207), will be crucial in the proof of Theorem 1.1.

Recall also the classical form of the Borel-Cantelli quantitative Lemma ([8], Theorem 3 or [12], Theorem 8.3.1).

**Lemma 2.3.** *Let $\{A_k, k \geq 1\}$ be a sequence of events satisfying*

$$\mathbb{P}(A_k \cap A_\ell) \leq \mathbb{P}(A_k)\mathbb{P}(A_\ell) + \gamma_{\ell-k}\mathbb{P}(A_\ell), \qquad (\forall \ell \geq k \geq 1)$$

*where $\gamma_i \geq 0$ and $\sum_{i=0}^\infty \gamma_i < \infty$. Let $\psi_n = \sum_{k=1}^n \mathbb{P}(A_k)$ and assume that $\psi_n \to \infty$ with $n$. Then for every $a > 3/2$,*

$$\sum_{k=1}^n \chi_{A_k} \overset{a.s.}{=} \psi_n + \mathcal{O}_a\left(\psi_n^{1/2}\left(\log\psi_n\right)^a\right).$$

We finally need a suitable invariance principle for sums of independent random variables. This one is due to Sakhanenko (see [10], Theorem 1). We give its most appropriate formulation for our purpose. Let $\{\xi_j, j \geq 1\}$ be independent centered random variables with absolute second moments. Let $t_k = \sum_{j=1}^k \mathbb{E}\xi_j^2$, $S_k = \sum_{j=1}^k \xi_j$ and let $\{r_k, k \geq 1\}$ be some non-decreasing sequence of positive reals. Let $\alpha \geq 2$, $y > 0$. Put successively,

$$\begin{aligned}
\Delta_n &= \sup_{k\leq n} |S_k - W(t_k)|, \\
\Delta &= \sup_{n\geq 1} \frac{\Delta_n}{r_n}, \\
\overline{\xi} &= \sup_{j\geq 1} \frac{|\xi_j|}{r_j}, \\
L_\alpha(y) &= \sum_{j\geq 1} \mathbb{E}\min\left\{\frac{|\xi_j|^\alpha}{y^\alpha r_j^\alpha}, \frac{|\xi_j|^2}{y^2 r_j^2}\right\}. \qquad (2.6)
\end{aligned}$$

**Lemma 2.4.** *There exists an absolute constant $C$ such that for any fixed $\alpha$, there exists a Brownian motion $W$ such that for all $x > 0$,*

$$\mathbb{P}\left\{\Delta \geq C\alpha x\right\} \leq L_\alpha(x).$$

Now we pass to the Kubilius model. Recall that $p$ denotes some arbitrary prime number. Let $\{Y_p, p \geq 1\}$ be a sequence of independent binomial random variables

such that $\mathbb{P}\{Y_p = 1\} = 1/p$ and $\mathbb{P}\{Y_p = 0\} = 1 - 1/p$. We can view $Y_p$ as modelling whether or not an integer taken at random is divisible by $p$. Let

$$T_n = \sum_{p \leq n} Y_p, \qquad S_n = T_n - \mathbb{E}T_n.$$

Then $\mathbb{E}S_n^2 = s_n^2 = \log\log n + \mathcal{O}(1)$ by (1.10). The sequence $\{T_n, n \geq 1\}$ is known to asymptotically behave as the truncated prime divisor function

$$\omega(m, t) = \#\{p \leq t : p|m\},$$

at least when $t$ is not too close to $m$. More precisely, let

$$\omega_r(m) = \big(\omega(m, 1), \ldots, \omega(m, r)\big),$$

where $r$ is some integer with $2 \leq r \leq x$, and put $u = \frac{\log x}{\log r}$. Then, given $c < 1$ arbitrary, we have *uniformly* in $x, r$ and $Q \subset \mathbb{Z}^r$,

$$\frac{\#\{m \leq x : \omega_r(m) \in Q\}}{x} = \mathbb{P}\big\{(T_1, \ldots, T_r) \in Q\big\} + \mathcal{O}\big(x^{-c} + e^{-u\log u}\big). \qquad (2.7)$$

See Lemmas 3.2, 3.5 in [3] Chapter 3. See also [11], Theorem 1 for a more precise result involving the Dickman function.

*Remark* 2.5. There are natural restrictions in the application of this estimate to *asymptotic* studies, due to the error term $e^{-u\log u}$. To make it small, it requires if $r = r(x)$ that $r(x) = \mathcal{O}_\varepsilon(x^\varepsilon)$ for all $\varepsilon > 0$. This amounts to truncate the prime divisor function $\omega(m)$ at level $\mathcal{O}_\varepsilon(x^\varepsilon)$, which is satisfactory as long as $m \ll x$. However, these integers have a neglectable contribution on the size of the left-term of (2.7). Therefore the model is mostly adapted to the analysis of the distribution of the small divisors of an integer. See [3] p.122, see also [11] (Introduction) for a complete and precise analysis of this point.

Estimate (2.7) can be for instance used to estimate the number of integers having no prime divisors in prescribed sets. Let $I = [p, q]$, $q \leq x$; as $\#\{m \leq x : p|m \Rightarrow p \notin I\} = \#\{m \leq x : \omega(m, p) = \ldots = \omega(m, q)\}$, it follows that

$$\frac{1}{x}\#\Big\{m \leq x : p|m \Rightarrow p \notin I\Big\} = \prod_{p \in I}\Big(1 - \frac{1}{p}\Big) + \mathcal{O}\big(x^{-c} + e^{-u\log u}\big). \quad (2.8)$$

Choosing $I = [2, y]$, next $I = [y, x]$ allows to recover known formula on the smallest or largest prime divisors of $m$.

Clearly, the approximation formula (2.7) can be used to transfer properties from $(T_k)$ to $\omega$. Let indeed $f$ be such that $f(N) = o_\rho(N^\rho)$. Recall that $I_N = \{j : N \leq s_j^2 \leq Nf(N)\}$ and let $\mathcal{N}$ be some fixed increasing sequence of reals. Moreover, let $M_i : \mathbb{N} \to \mathbb{R}^+$ be non-decreasing with $\lim_{x\to\infty} M_i(x) = \infty$, $i = 1, 2$, and such that

$$1 \leq M_1(x) < M_2(x), \qquad M_2(x) = \mathcal{O}_\varepsilon(x^\varepsilon). \qquad (2.9)$$

Let $r = r(x) \sim M_2(x)f(M_2(x))$, $r$ integer. Then

$$u = u(x) = \frac{\log x}{\log r(x)} \sim \frac{\log x}{\log M_2(x)f(M_2(x))} \to \infty$$

with $x$. Put

$$Q_x = \bigcup_{\substack{M_1(x) < N \le M_2(x) \\ N \in \mathcal{N}}} \left\{ (\nu_1, \dots, \nu_r) \in \mathbb{Z}^r : \sup_{j \in I_N} \frac{|\nu_j - s_j^2|}{s_j} \le z \right\}.$$

By applying (2.7) with $Q = Q_x$, we get the useful comparison relation

**Lemma 2.6.** *For any $z > 0$, as $x$ tends to infinity,*

$$\frac{1}{x} \# \left\{ m \le x : \inf_{\substack{M_1(x) < N \le M_2(x) \\ N \in \mathcal{N}}} \sup_{j \in I_N} \frac{|\omega(m, j) - \log \log j|}{\sqrt{\log \log j}} \le z \right\}$$

$$= \mathbb{P} \left\{ \inf_{\substack{M_1(x) < N \le M_2(x) \\ N \in \mathcal{N}}} \sup_{j \in I_N} \frac{|T_j - \log \log j|}{\sqrt{\log \log j}} \le z \right\} + o(1).$$

## 3. Proof of Theorem 1.1

By stationarity and by using (2.4),

$$\frac{K_1(z)}{f(e^k)^{\lambda(z)}} \le \mathbb{P}\big(A_k(f, z)\big) = \mathbb{P} \left\{ \sup_{0 \le s \le \log f(e^k)} |U(s)| < z \right\} \le \frac{K_2(z)}{f(e^k)^{\lambda(z)}}. \tag{3.1}$$

By summing up,

$$K_1(z) \sum_{k=1}^n f(e^k)^{-\lambda(z)} \le \nu_n(f, z) \le K_2(z) \sum_{k=1}^n f(e^k)^{-\lambda(z)}. \tag{3.2}$$

If the series $\Sigma(f) = \sum_k f(e^k)^{-\lambda(z)}$ converges, by the first Borel-Cantelli lemma

$$\mathbb{P} \left\{ \sup_{k \le s \le k + \log f(e^k)} |U(s)| > z, \qquad k \text{ eventually} \right\} = 1.$$

Hence $f \in \mathcal{U}_z$. Now consider the case $\Sigma(f) = \infty$. We shall prove that $f \in \mathcal{V}_z$. Let $0 < c_1 < 1/\lambda(z) < c_2$ and put

$$f_1(t) = \log^{c_1} t, \qquad f_2(t) = \log^{c_2} t.$$

We may assume $f_1 \le f \le f_2$. This is a standard device. Indeed, as $f_2 \in \mathcal{U}_z$, we have the implication: $(f_1 \vee f) \wedge f_2 \in \mathcal{V}_z \Rightarrow (f_1 \vee f) \in \mathcal{V}_z \Rightarrow f \in \mathcal{V}_z$. So it suffices to prove that $(f_1 \vee f) \wedge f_2 \in \mathcal{V}_z$. We now use the simplified notation $A_k(f, z) = A_k$ and notice that $K'(z)k^{-c_2\lambda(z)} \le \mathbb{P}(A_k) \le K''(z)k^{-c_1\lambda(z)}$. By Lemma 2.2, for every $\ell > k$,

$$\frac{\big|\mathbb{P}(A_k \cap A_\ell) - \mathbb{P}(A_k)\mathbb{P}(A_\ell)\big|}{\sqrt{\mathbb{P}(A_k)(1 - \mathbb{P}(A_k))\mathbb{P}(A_\ell)(1 - \mathbb{P}(A_\ell))}} \le C_1 e^{-C_2(\ell - k)},$$

$C_1, C_2$ being absolute constants. Hence

$$\begin{aligned}
\big|\mathbb{P}(A_k \cap A_\ell) - \mathbb{P}(A_k)\mathbb{P}(A_\ell)\big| &\le C_1 e^{-C_2(\ell - k)} \sqrt{\mathbb{P}(A_k)\mathbb{P}(A_\ell)} \\
&\le C_1 e^{-C_2(\ell - k)} \mathbb{P}(A_\ell) \Big(\frac{\mathbb{P}(A_k)}{\mathbb{P}(A_\ell)}\Big)^{1/2} \\
&\le C(z)\mathbb{P}(A_\ell) e^{-C_2(\ell - k)} \Big(\frac{\ell^{c_2}}{k^{c_1}}\Big)^{\lambda(z)/2}.
\end{aligned}$$

But for some absolute constant $C_3 < C_2$ and $C_4 > 0$ depending on $z$, we have

$$e^{-C_2(\ell-k)}\left(\frac{\ell^{c_2}}{k^{c_1}}\right)^{\lambda(z)/2} \leq C_4 e^{-C_3(\ell-k)}.$$

Indeed let $\ell = (H+1)k$, $H \geq 0$. This amounts to show that

$$(H+1)^{c_2\lambda(z)/2}k^{(c_2-c_1)\lambda(z)/2} \leq C_4 e^{(C_2-C_3)Hk}.$$

We use the following inequality. Let $\delta, \beta, \varepsilon$ be positive reals with $\delta \geq \beta$. Then there exists $C$ depending on $\delta, \beta, \varepsilon$ only such that $H^\delta k^\beta \leq Ce^{\varepsilon Hk}$ for all non-negative reals $H, k$ with $k \geq 1$. Indeed, if $0 \leq H \leq 1$, then $H^\delta k^\beta \leq (Hk)^\beta \leq Ce^{\varepsilon Hk}$. And if $H > 1$, $H^\delta k^\beta \leq (Hk)^\delta \leq Ce^{\varepsilon Hk}$.

Applying this with $\delta = c_2\lambda(z)/2$, $\beta = (c_2-c_1)\lambda(z)/2$ yields

$$H^{c_2\lambda(z)/2}k^{(c_2-c_1)\lambda(z)/2} \leq Ce^{\varepsilon Hk},$$

which implies our claim. Thereby

$$\left|\mathbb{P}(A_k \cap A_\ell) - \mathbb{P}(A_k)\mathbb{P}(A_\ell)\right| \leq C_4 e^{-C_3(k-\ell)}\mathbb{P}(A_\ell).$$

Lemma 2.3 thus applies, and we deduce (for every $a > 3/2$),

$$\sum_{k=1}^n \chi_{A_k} \overset{a.s.}{=} \nu_n(f,z) + \mathcal{O}_a\left(\nu_n(f,z)^{1/2}\log^a \nu_n(f,z)\right).$$

In particular

$$\mathbb{P}\left\{\sup_{k \leq s \leq k+\log f(e^k)}|U(s)| \geq z, \qquad k \text{ infinitely often}\right\} = 1.$$

Hence also $f \in \mathcal{V}_z$.

Corollary 1.2 follows easily. Indeed, let $0 < c \leq 1/\lambda(z)$. By Theorem 1.1, $N_n(f_c, z) \uparrow \infty$ almost surely. And so $\mathbb{P}\{J(f_c) \leq z\} = 1$. Now if $c > 1/\lambda(z)$, in view of estimate (2.4) the series $\sum_{k=1}^\infty \mathbb{P}\{A_k(f_c, z)\}$ converges. And by the first Borel-Cantelli lemma $\mathbb{P}\{J(f_c) > z\} = 1$. Corollary 1.3 is just a reformulation of Corollary 1.2 using the variable change $s = e^t$.

## 4. Proof of Theorem 1.6

Now we can pass to the proof. Let $\varepsilon, \eta$ be positive reals. Let $\alpha$ sufficiently large so that $\varepsilon\alpha > 1 + \eta$. Apply Lemma 2.4 to $S_n$ (here $\xi_p = Y_p - \mathbb{E}Y_p$). Choose $r_p = (\log\log p)^{\frac{1+\eta}{\alpha}}$ and recall that $\mathbb{E}|\xi_p|^\alpha \sim 1/p$ for $p$ large. Then

$$\sum_p \frac{\mathbb{E}|\xi_p|^\alpha}{r_p^\alpha} \leq C\sum_p \frac{1}{p(\log\log p)^{1+\eta}} \leq C\sum_j \frac{1}{j\log j(\log\log j)^{1+\eta}} < \infty.$$

We have used the fact that if $p_j$ denotes the $j$-th prime number in the increasing order, then $p_j \sim j\log j$. Now notice the following simple estimate valid for all positive $y$,

$$L_\alpha(y) \leq y^{-\alpha}\sum_{j\geq 1}\frac{\mathbb{E}|\xi_j|^\alpha}{r_j^\alpha}.$$

We deduce that $L_\alpha(y) \le C_\alpha y^{-\alpha}$. Recall that $\mathbb{E}S_n^2 = s_n^2$. Therefore there exists a Brownian motion $W$ such that if

$$\mathbf{\Upsilon}_\varepsilon = \sup_{n \ge 1} \frac{\sup_{j \le n} |S_j - W(s_j^2)|}{(\log \log n)^\varepsilon}, \tag{4.1}$$

then $\mathbb{E}\mathbf{\Upsilon}_\varepsilon^\beta < \infty$, $\beta < \alpha$. We will just use the fact that $\mathbb{E}\mathbf{\Upsilon}_\varepsilon < \infty$. Let $z' > z$. By using Lemma 2.6, we have

$$\frac{1}{x}\#\Big\{ m \le x : \inf_{\substack{M_1(x) < N \le M_2(x) \\ N \in \mathcal{N}}} \sup_{j \in I_N} \frac{|\omega(m, j) - s_j^2|}{s_j} \le z \Big\}$$

$$= \mathbb{P}\Big\{ \inf_{\substack{M_1(x) < N \le M_2(x) \\ N \in \mathcal{N}}} \sup_{j \in I_N} \frac{|S_j|}{s_j} \le z \Big\} + o(1)$$

$$\le \mathbb{P}\Big\{ \inf_{\substack{M_1(x) < N \le M_2(x) \\ N \in \mathcal{N}}} \sup_{j \in I_N} \frac{|W(s_j^2)|}{s_j} \le z' \Big\} + \mathbb{P}\{A\} + o(1), \tag{4.2}$$

where we set

$$A = \Big\{ \inf_{\substack{M_1(x) < N \le M_2(x) \\ N \in \mathcal{N}}} \sup_{j \in I_N} \frac{|W(s_j^2)|}{s_j} > z', \ \inf_{\substack{M_1(x) < N \le M_2(x) \\ N \in \mathcal{N}}} \sup_{j \in I_N} \frac{|S_j|}{s_j} \le z \Big\}.$$

We have

$$\Big| \inf_{\substack{M_1(x) < N \le M_2(x) \\ N \in \mathcal{N}}} \sup_{j \in I_N} \frac{|W(s_j^2)|}{s_j} - \inf_{\substack{M_1(x) < N \le M_2(x) \\ N \in \mathcal{N}}} \sup_{j \in I_N} \frac{|S_j|}{s_j} \Big|$$

$$\le \sup_{\substack{M_1(x) < N \le M_2(x) \\ N \in \mathcal{N}}} \sup_{j \in I_N} \frac{|S_j - W(s_j^2)|}{s_j} \le \sup_{M_1(x) < N \le M_2(x)} \sup_{j \le j^*} \frac{|S_j - W(s_j^2)|}{s_j},$$

where $j^*$ denote the largest indice such that $s_j^2 \in I_N$ of $I_N$. Thus

$$\mathbb{P}\{A\} \le \mathbb{P}\Big\{ \sup_{M_1(x) < N \le M_2(x)} \sup_{j \le j^*} \frac{|S_j - W(s_j^2)|}{s_j} > z' - z \Big\}. \tag{4.3}$$

Let $\varepsilon' > \varepsilon$. Since $f(N) = o_\rho(N^\rho)$ by assumption and $s_j^2 \sim \log \log j$ by (1.10), we have for all $N$ sufficiently large, $N \ge N(\varepsilon, \varepsilon')$ say,

$$\sup_{j \le j^*} |S_j - W(s_j^2)| \le \mathbf{\Upsilon}_\varepsilon (\log \log j^*)^\varepsilon \le C\mathbf{\Upsilon}_\varepsilon (Nf(N))^\varepsilon \le C\mathbf{\Upsilon}_\varepsilon N^{\varepsilon'}.$$

Then

$$\sup_{j \le j^*} \frac{|S_j - W(s_j^2)|}{s_j} \le CN^{-\frac{1}{2}} \sup_{j \le j^*} |S_j - W(s_j^2)| \le C\mathbf{\Upsilon}_\varepsilon N^{-\frac{1}{2}+\varepsilon'}. \tag{4.4}$$

Thereby for $N \ge N(\varepsilon, \varepsilon')$,

$$\sup_{M_1(x) < N \le M_2(x)} \sup_{j \le j^*} \frac{|S_j - W(s_j^2)|}{s_j} \le C\mathbf{\Upsilon}_\varepsilon M_1(x)^{-\frac{1}{2}+\varepsilon'}. \tag{4.5}$$

It follows that

$$\mathbb{P}\{A\} \le \mathbb{P}\Big\{ \mathbf{\Upsilon}_\varepsilon > C(z' - z)M_1(x)^{\frac{1}{2}-\varepsilon'} \Big\} \le \frac{C}{(z' - z)M_1(x)^{\frac{1}{2}-\varepsilon'}} \mathbb{E}\mathbf{\Upsilon}_\varepsilon.$$

Consequently $\mathbb{P}\{A\} = o(x)$, and we deduce from (4.2) that

$$\frac{1}{x}\#\Big\{m \le x : \inf_{\substack{M_1(x) < N \le M_2(x) \\ N \in \mathcal{N}}} \sup_{j \in I_N} \frac{|\omega(m,j) - s_j^2|}{s_j} \le z\Big\}$$

$$\le \mathbb{P}\Big\{\inf_{\substack{M_1(x) < N \le M_2(x) \\ N \in \mathcal{N}}} \sup_{j \in I_N} \frac{|W(s_j^2)|}{s_j} \le z'\Big\} + o(1). \qquad (4.6)$$

Now let $0 < z'' < z$. As

$$\mathbb{P}\Big\{\inf_{\substack{M_1(x) < N \le M_2(x) \\ N \in \mathcal{N}}} \sup_{j \in I_N} \frac{|W(s_j^2)|}{s_j} \le z'', \sup_{\substack{M_1(x) < N \le M_2(x) \\ N \in \mathcal{N}}} \sup_{j \in I_N} \frac{|S_j - W(s_j^2)|}{s_j} \le z - z''\Big\}.$$

$$\le \mathbb{P}\Big\{\inf_{\substack{M_1(x) < N \le M_2(x) \\ N \in \mathcal{N}}} \sup_{j \in I_N} \frac{|S_j|}{s_j} \le z\Big\},$$

we deduce from Lemma 2.6

$$\frac{1}{x}\#\Big\{m \le x : \inf_{\substack{M_1(x) < N \le M_2(x) \\ N \in \mathcal{N}}} \sup_{j \in I_N} \frac{|\omega(m,j) - s_j^2|}{s_j} \le z\Big\}$$

$$= \mathbb{P}\Big\{\inf_{\substack{M_1(x) < N \le M_2(x) \\ N \in \mathcal{N}}} \sup_{j \in I_N} \frac{|S_j|}{s_j} \le z\Big\} + o(1)$$

$$\ge \mathbb{P}\Big\{\inf_{\substack{M_1(x) < N \le M_2(x) \\ N \in \mathcal{N}}} \sup_{j \in I_N} \frac{|W(s_j^2)|}{s_j} \le z''\Big\} - \mathbb{P}\{B\} + o(1), \qquad (4.7)$$

where we set

$$B = \Big\{\inf_{\substack{M_1(x) < N \le M_2(x) \\ N \in \mathcal{N}}} \sup_{j \in I_N} \frac{|W(s_j^2)|}{s_j} \le z'', \sup_{\substack{M_1(x) < N \le M_2(x) \\ N \in \mathcal{N}}} \sup_{j \in I_N} \frac{|S_j - W(s_j^2)|}{s_j} > z - z''\Big\}.$$

By operating similarly, we also get

$$\frac{1}{x}\#\Big\{m \le x : \inf_{\substack{M_1(x) < N \le M_2(x) \\ N \in \mathcal{N}}} \sup_{j \in I_N} \frac{|\omega(m,j) - s_j^2|}{s_j} \le z\Big\}$$

$$\ge \mathbb{P}\Big\{\inf_{\substack{M_1(x) < N \le M_2(x) \\ N \in \mathcal{N}}} \sup_{j \in I_N} \frac{|W(s_j^2)|}{s_j} \le z''\Big\} + o(1). \qquad (4.8)$$

The proof is now complete.

*Remark* 4.1. It follows from (4.4) that for all $0 < \delta < 1/2$ and $d \ge 0$

$$\mathbb{E} \sup_N \frac{1}{N^{-\delta}} \sup_{j \in I_N} \Big|\frac{S_j - W(s_j^2)}{s_j}\Big|^d < \infty. \qquad (4.9)$$

Consequently for any increasing unbounded sequence of reals $\mathcal{N}$,

$$\liminf_{k \to \infty} \sup_{j \in I_{N_k}} \frac{|S_j|}{\sqrt{\log \log j}} \overset{a.s.}{=} \liminf_{k \to \infty} \sup_{j \in I_{N_k}} \frac{|W(s_j^2)|}{\sqrt{\log \log j}}. \qquad (4.10)$$

## 5. Proof of Corollary 1.7

Let $z'' < z$. Let $f = f_c$ with $c < 1/\lambda(z'')$, $\mathcal{N} = \{e^k, k \geq 1\}$. Let also $0 < \gamma < 1$. Observe that

$$
\mathbb{P}\Big\{ \inf_{M_1(x) < N = e^k \leq M_2(x)} \sup_{j \in I_N} \frac{|W(s_j^2)|}{s_j} \leq z'' \Big\}
$$

$$
\geq \ \mathbb{P}\Big\{ \inf_{M_1(x) < N = e^k \leq M_2(x)} \sup_{e^k \leq t \leq e^k k^c} \frac{|W(t)|}{\sqrt{t}} \leq z'' \Big\}
$$

$$
\geq \ \mathbb{P}\Big\{ \sum_{\log M_1(x) < k \leq \log M_2(x)} \chi\Big\{ \sup_{e^k \leq t \leq e^k k^c} \frac{|W(t)|}{\sqrt{t}} \leq z'' \Big\} > 0 \Big\}
$$

$$
= \ \mathbb{P}\Big\{ \sum_{\log M_1(x) < k \leq \log M_2(x)} \chi\{A_k(z'')\} > 0 \Big\}
$$

$$
\geq \ \mathbb{P}\Big\{ \sum_{\log M_1(x) < k \leq \log M_2(x)} \chi\{A_k(z'')\} \geq \gamma \sum_{\log M_1(x) < k \leq \log M_2(x)} \mathbb{P}\{A_k(z'')\} \Big\}.
$$

Thus

$$
\mathbb{P}\Big\{ \inf_{M_1(x) < N = e^k \leq M_2(x)} \sup_{j \in I_N} \frac{|W(s_j^2)|}{s_j} \leq z'' \Big\}
$$

$$
= \mathbb{P}\Big\{ \frac{N_{\log M_2(x)}(f_c, z'') - N_{\log M_1(x)}(f_c, z'')}{\nu_{\log M_2(x)}(f_c, z'') - \nu_{\log M_1(x)}(f_c, z'')} \geq \gamma \Big\}. \tag{5.1}
$$

By Corollary 1.2,

$$
\lim_{n \to \infty} \frac{N_n(f_c, z'')}{\nu_n(f_c, z'')} \overset{a.s.}{=} 1 \quad \text{and} \quad K_1(z'') \leq \frac{\nu_n(f_c, z)}{n^{1 - c\lambda(z'')}} \leq K_2(z'').
$$

By assumption, we have $\log M_1(x) = o\big(\log M_2(x)\big)$. Thus $\nu_{\log M_1(x)}(f_c, z'') = o\big(\nu_{\log M_2(x)}(f_c, z'')\big)$. And it follows that

$$
\lim_{n \to \infty} \frac{N_{\log M_2(x)}(f_c, z'') - N_{\log M_1(x)}(f_c, z'')\}}{\nu_{\log M_2(x)}(f_c, z'') - \nu_{\log M_1(x)}(f_c, z'')} \overset{a.s.}{=} 1.
$$

Consequently

$$
\liminf_{x \to \infty} \mathbb{P}\Big\{ \frac{N_{\log M_2(x)}(f_c, z'') - N_{\log M_1(x)}(f_c, z'')\}}{\nu_{\log M_2(x)}(f_c, z'') - \nu_{\log M_1(x)}(f_c, z'')} \geq \gamma \Big\} = 1.
$$

By combining this with (5.1), we get

$$
\liminf_{x \to \infty} \mathbb{P}\Big\{ \inf_{M_1(x) < N = e^k \leq M_2(x)} \sup_{j \in I_N} \frac{|W(s_j^2)|}{s_j} \leq z'' \Big\} = 1
$$

In view of Theorem 1.6, this also implies

$$
\lim_{x \to \infty} \frac{1}{x} \#\Big\{ m \leq x : \inf_{\log M_1(x) < k \leq \log M_2(x)} \sup_{e^k \leq s_j^2 \leq e^k k^c} \frac{|\omega(m, j) - s_j^2|}{s_j} \leq z \Big\} = 1.
$$

The proof is now complete.

## 6. Proof of Theorem 1.10

The sets $A_k(c, z)$ being introduced before Theorem 1.1, we also define

$$
\begin{aligned}
B_k(c, z) &= \left\{ \sup_{j \in I_{N_k}} \frac{|\omega(m, j) - s_j|}{s_j} \leq z \right\} \\
C_k(c, z) &= \left\{ \sup_{j \in I_{N_k}} \frac{|S_j|}{s_j} \leq z \right\} \\
D_k(c, z) &= \left\{ \sup_{j \in I_{N_k}} \frac{|W(s_j^2)|}{s_j} \leq z \right\}.
\end{aligned}
$$

Fix $u > 0$ and let $\eta > 0$. By (4.9), on a measurable set of full measure, we have for all $k$ large enough, $D_k(c, u) \subseteq C_k(c, u + \eta)$. Let $0 < c < 1/\lambda(z)$. By Theorem 1.1,

$$
\lim_{n \to \infty} \frac{\sum_{k=1}^n \chi_{A_k(c, z)}}{\sum_{k=1}^n \mathbb{P}(A_k(c, z))} \overset{a.s.}{=} 1.
$$

Let $0 \leq M_2(x) \uparrow \infty$ $x$ and such that $M_2(x) = \mathcal{O}_\varepsilon(x^\varepsilon)$. Obviously,

$$
\lim_{x \to \infty} \inf_{M_1(x) \leq n \leq M_2(x)} \frac{\sum_{k=1}^n \chi_{A_k(c, z)}}{\sum_{k=1}^n \mathbb{P}(A_k(c, z))} \overset{a.s.}{=} 1.
$$

Further

$$
\kappa_1 \leq \frac{\sum_{k=1}^n \mathbb{P}(A_k(c, z))}{n^{1 - c\lambda(z)}} \leq \kappa_2,
$$

for some positive constants $\kappa_1, \kappa_2$. Let $z' > z$. Since $A_k(c, z) \subseteq D_k(c, z')$, it follows that with probability one

$$
\begin{aligned}
1 \overset{a.s.}{=} & \lim_{x \to \infty} \inf_{M_1(x) \leq n \leq M_2(x)} \frac{\sum_{k=1}^n \chi_{A_k(c, z)}}{\sum_{k=1}^n \mathbb{P}(A_k(c, z))} \\
\leq & \limsup_{x \to \infty} \inf_{M_1(x) \leq n \leq M_2(x)} \frac{\sum_{k=1}^n \chi\{C_k(c, z')\}}{\sum_{k=1}^n \mathbb{P}(A_k(c, z))}.
\end{aligned}
$$

Let $0 < \varepsilon < 1$ and put

$$
Q_x = \Bigg\{ (\nu_1, \ldots, \nu_r) \in \mathbb{Z}^r :
$$

$$
\inf_{M_1(x) \leq n \leq M_2(x)} \frac{1}{\sum_{k=1}^n \mathbb{P}(C_k(c, z'))} \sum_{k=1}^n \chi\left\{ \sup_{j \in I_{N_k}} \frac{|\nu_j - s_j^2|}{s_j} \leq z' \right\} \leq \varepsilon \Bigg\}.
$$

By applying (2.7) with $Q = Q_x$, we get

$$
\frac{1}{x} \# \left\{ m \leq x : \inf_{M_1(x) \leq n \leq M(x)} \frac{1}{\sum_{k=1}^n \mathbb{P}(C_k(c, z'))} \sum_{k=1}^n \chi\{B_k(c, z')\} \leq \varepsilon \right\}
$$

$$
= \mathbb{P} \left\{ \inf_{M_1(x) \leq n \leq M(x)} \frac{1}{\sum_{k=1}^n \mathbb{P}(C_k(c, z'))} \sum_{k=1}^n \chi\{C_k(c, z')\} \leq \varepsilon \right\} + o(1).
$$

Thus

$$
\limsup_{x \to \infty} \frac{1}{x} \# \left\{ m \leq x : \inf_{M_1(x) \leq n \leq M(x)} \frac{1}{\sum_{k=1}^n \mathbb{P}(C_k(c, z'))} \sum_{k=1}^n \chi\{B_k(c, z')\} \leq \varepsilon \right\}
$$

$$= \limsup_{x \to \infty} \mathbb{P} \Big\{ \inf_{M_1(x) \le n \le M(x)} \frac{1}{\sum_{k=1}^n \mathbb{P}(C_k(c, z'))} \sum_{k=1}^n \chi\{C_k(c, z')\} \le \varepsilon \Big\} = 0.$$

This being true for all $0 < \varepsilon < 1$, we infer that

$$\lim_{x \to \infty} \frac{1}{x} \# \Big\{ m \le x : \inf_{M_1(x) \le n \le M(x)} \frac{1}{\sum_{k=1}^n \mathbb{P}(C_k(c, z'))} \sum_{k=1}^n \chi\{B_k(c, z')\} \ge 1 \Big\} = 1.$$

Finally, for some $\kappa > 0$ depending on $z$,

$$\lim_{x \to \infty} \frac{1}{x} \# \Big\{ m \le x : \inf_{M_1(x) \le n \le M(x)} \frac{1}{n^{1 - c\lambda(z)}} \sum_{k=1}^n \chi\{B_k(c, z')\} \ge \kappa \Big\} = 1.$$

## 7. Proof of Theorem 1.5

Let $1/\alpha < \beta < 1/2$. Take $r_n = (\sum_{i=1}^j \mathbb{E}|X_i|^2)^\beta = z_n^{2\beta}$. We notice that

$$\sum_{j \ge 1} \frac{\mathbb{E}|X_j|^\alpha}{r_j^\alpha} = \sum_{j \ge 1} \frac{\mathbb{E}|X_j|^\alpha}{(\sum_{i=1}^j \mathbb{E}|X_i|^2)^{\alpha\beta}} \le C \sum_{j \ge 1} \frac{\mathbb{E}|X_j|^\alpha}{(\sum_{i=1}^j \mathbb{E}|X_i|^\alpha)^{\alpha\beta}} < \infty,$$

since $\alpha\beta > 1$. Thus

$$L_\alpha(y) \le y^{-\alpha} \sum_{j \ge 1} \mathbb{E} \frac{|X_j|^\alpha}{r_j^\alpha} \le C y^{-\alpha}.$$

By Lemma 2.4, there exists a Brownian motion $W$ such that if

$$\Upsilon = \sup_n \frac{1}{r_n} \sup_{j \le n} |Z_j - W(z_j^2)|$$

then $\mathbb{E}\Upsilon^{\alpha'} < \infty$, $\alpha' < \alpha$. Now let $j_p^* = \max\{j : r_j \le 2^p\}$. As

$$\sup_{2^{p-1} < r_j \le 2^p} \frac{|Z_j - W(z_j^2)|}{r_j} \le \frac{2}{r_{j_p^*}} \sup_{j \le j_p^*} |Z_j - W(z_j^2)|,$$

whenever $\{j : 2^{p-1} < r_j \le 2^p\} \ne \emptyset$, it follows that

$$\sup_{r_j \ge 1} \frac{|Z_j - W(z_j^2)|}{r_j} \le 2\Upsilon.$$

Let $j(N) = \max(J_N)$. Hence

$$\begin{aligned}
\Big| \sup_{j \in J_N} \frac{|Z_j|}{z_j} - \sup_{j \in J_N} \frac{|W(z_j^2)|}{z_j} \Big| &\le \sup_{j \in J_N} \frac{|Z_j - |W(z_j^2)||}{z_j} = \sup_{j \in J_N} \frac{|Z_j - |W(z_j^2)||}{z_j^{1-2\beta} z_j^{2\beta}} \\
&\le \Big( \sup_{j \in J_N} \frac{1}{z_j^{1-2\beta}} \Big) \sup_{j \le j(N)} \frac{|Z_j - |W(z_j^2)||}{z_j^{2\beta}} \\
&\le \Big( \sup_{j \in J_N} \frac{1}{z_j^{1-2\beta}} \Big) \Upsilon \\
&\to 0,
\end{aligned}$$

as $N \to \infty$ almost surely, since $\beta < 1/2$. By specifying this for $N = e^k$, we therefore deduce

$$\liminf_{k \to \infty} \sup_{e^k \leq z_j^2 \leq e^k f(e^k)} \frac{|Z_j|}{z_j} \overset{a.s.}{=} \liminf_{k \to \infty} \sup_{e^k \leq z_j^2 \leq e^k f(e^k)} \frac{|W(z_j^2)|}{s_j}.$$

almost surely. This together with Corollary 1.2 allows to conclude.

## 8. Concluding Remarks

Clearly, the approximation formula (2.7) applies to strongly additive arithmetic functions $f(n) = \sum_{p|n} f(p)$, and associated truncated functions. For additive arithmetic functions $f(n) = \sum_{p^\nu || n} f(p)$, the comparizon is made with the sums of independent random variables $\xi_p$ defined by $\mathbb{P}\{\xi_p = f(p^\nu)\} = (1 - 1/p)p^{-\nu}$, $\nu = 0, 1, \ldots$. See [3], [11]. Special cases will be investigated elsewhere.

## References

[1] Csáki E.: Some limit theorems for empirical processes, in: Vilaplana, J.P., Puri, M.I. (Eds), *Recent Advances in Statistics and Probability* (Proc. **4th** IMSIBAC) VSP, Utrecht (1994) 247–254.

[2] Csáki E., Shi Z.: Some liminf results for two-parameter processes, *Stoch. Proc.&Their Appl.* **78** (1998) 27-46.

[3] Elliott, P.D.T.A.: *Probabilistic Number Theory-I*, Springer Verlag, Berlin-New-York, 1979.

[4] Ford K., Tenenbaum G.: Localized large sums of random variables, *Stat.&Prob. Letters* **78** (2008) 84–89.

[5] Ibragimov I.A., Rozanov Y.A.: *Gaussian Random processes*, 1974.

[6] Kolmogorov A.N., Rozanov Y.A.: On a strong mixing condition for stationary random Gaussian processes, *Teoria Verojatn. Primen.* **5** (1960) 222–227.

[7] Newell G.F.: Asymptotic extreme value distributions for one dimensional diffusion processes, *J. Math. Mech.* **11** No1 (1962) 481–496.

[8] Philipp, W.: Some metric theorems in number theory, *Pacific J. Math.* **20** (1967) 109–127.

[9] Sakhanenko A.I.: Estimates in the invariance principle, in: Trudy Inst. Mat. SO AN SSSR *Nauka Novosibirsk* **5** (1985) 27–44. (Russian)

[10] Sakhanenko A.I.: Estimates in the invariance principle in terms of truncated power moments, *Siberian Math. Journal* **47** no6 (2006) 1113-1127.

[11] Tenenbaum G.: Crible d'Eratosthène et modèle de Kubilius, in: Gyóry, Iwaniec, Urbanowicz (Eds), *Number Theory in Progress, Proceedings of the Conference in honor of Andrzej Schinzel*, Zakopane, Poland 1997, W. de Gruyter, Berlin-New-York (1999) 1099–1129.

[12] Weber M.: *Dynamical Systems and Processes*, European Mathematical Society Publishing House, IRMA Lectures in Mathematics and Theoretical Physics **14** xiii+761p, 2009.

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